**MATH 301 Week 2 Homework, Fall, 2015. Due Monday, September 21.**

**NAME:** \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

**Instructions:** You are encouraged to discuss homework, use online resources, and to seek help from the instructor when you need it, but your submitted write-up of your work must be your own, in your own words.

**Induction**

**#1**. Prove by induction that 8 + 11 + 14 + … + (3*n* + 5) = *n*(3*n* + 13)/2 for all *n* ∈ **N**.

**#2.** Prove the summation formula for the following finite geometric series.

Prove by induction that $\frac{3}{4}+ 3\left(\frac{1}{4}\right)^{2}+\cdots + 3\left(\frac{1}{4}\right)^{n}=1-\frac{1}{4^{n}}$ for every *n* ∈ **N.**

**#3**. Prove by induction that (1.2)*n* ≥ 1 + 0.2*n* for every *n* ∈ **N** .

**Relations and Functions**

**#4**. Find an example of a relation that is reflexive and symmetric, but not transitive. (Note that you can just list an appropriate set of ordered pairs.)

**#5**. Let *S* be a set. Let *A* and *B* be subsets of *S*. Let *R* be the relation given by *A R B* iff *A* ⊆ *B*.

 That is, *A* is related to *B* iff *A* is a subset of *B*.

State which of the three properties (reflexive, symmetric, and transitive) apply. If a property does *not* hold, provide a counterexample.

**#6.** State the range of each function *f*: **R** → **R**. (no work required to be shown)

 **#6(a)** *f*(*x*) = (*x* + 5)2 − 3

 **#6(b)** *f*(*x*) = 8 sin(6*x*)

**#7.** Let *X* = {1, 2, 3, 4} and  *Y* = {4, 5 ,6, 7}. Define a function *f*: *X* → *Y* as follows:

 *f*(1) = 5,  *f*(2) = 7,  *f*(3) = 6,  *f*(4) = 5.

For each statement, is it True or False? (no explanation required)

  \_\_\_\_\_\_\_\_ (a)  ∃ x ∈ *X* such that  *f*(*x*) = 2*x*.

  \_\_\_\_\_\_\_\_ (b)   ∀*y* ∈ *Y*, ∃ *x*∈ *X* such that  *f*(*x*) = *y.*

 \_\_\_\_\_\_\_\_ (c)  ∀*x*∈ *X*, ∃ *y* ∈ *Y* such that  *f (x*) = *y.*

  \_\_\_\_\_\_\_\_ (d)   ∃ *y*∈ *Y* such that ∀ *x* ∈ *X*,  *f*(*x*) = *y.*

  \_\_\_\_\_\_\_\_ (e)   *f*is injective.

 \_\_\_\_\_\_\_\_ (f)   *f*is surjective.

**#8.** Let *f*: *A* → *B*. Recall that *f* is injective iff **∀ *a*, *a'* ∈ *A*, if *f*(*a*) = *f*(*a'*), then *a* = *a'*.**

 Write the contrapositive of the quantified if-then statement (the statement in bold).

**#9**. **Claim**: The function *f*: **R** → **R** defined by *f*(*x*) = 8 − 5*x* is injective. Consider the following "proofs" of the claim. **INSTRUCTIONS: Critique each proof (A, B, C, D, E). For each proof, is it a valid argument establishing the claim or not? What are the flaws, if any?**

**Proof A:**

Let *a*, *a'* ∈ **R** and suppose *a* = *a'*.

Multiply both sides by −5, so −5*a* = −5*a'*

Add 8 to both sides, so 8 − 5*a* = 8 − 5*a'*

Thus, *f*(*a*) = *f*(*a'*).

Therefore *f* is injective.

**Proof B:**

 Let *a*, *a'* ∈ **R** and suppose *a* ≠ *a'*.

Then −5*a* ≠ −5*a'*

and also 8 − 5*a* ≠ 8 − 5*a'*

 So, *f*(*a*) ≠ *f*(*a'*).

Therefore *f* is injective.

**Proof C:**

Let *a* = 0 and *a'* = 1.

 *f*(*a*) = 8 and *f*(*a'*) = 3.

Since *f*(*a*) ≠ *f*(*a'* ) and *a* ≠ *a'* , *f* is injective.

**Proof D:**

Let *a*, *a'*∈ **R** and suppose *f*(*a*) = *f*(*a'*).

 Then 8 − 5*a* = 8 − 5*a'*.

Subtract 8 from both sides, so −5*a* = −5*a'* .

Divide both sides by −5, so *a* = *a'.*

Thus *f* is injective.

**Proof E:**

 Let *a*, *a'* ∈ **R** and suppose *f*(*a*) ≠ *f*(*a'*).

Then 8 − 3*a* ≠ 8 − 3*a'*

and −3*a* ≠ −3*a'*

and so *a* ≠ *a'*

Thus *f* is injective.

**10**. Define *f*: **R** → **R** defined by *f*(*x*) = *x*2. Let *S* = [0, 4] and *T* = [− 4, 0].

 **#10(a)** Find *f*(*S*), *f*(*T*), and *f*(*S* ∩ *T*). Is *f*(*S* ∩ *T*) = *f*(*S*) ∩ *f*(*T*)?

 **#10(b)** Find *f* −1(*S*), *f* −1 (*T*), and *f* −1 (*S* ∩ *T*). Is *f* −1 (*S* ∩ *T*) = *f* −1 (*S*) ∩ *f* −1 (*T*)?

**#11.** Classify each function as injective, surjective, bijective, or none of these, as appropriate. If *not* injective, briefly explain. If *not* surjective, briefly explain. (Otherwise, no explanation required.)

 **#11** (a) *f*: **Z** → **Z** defined by *f*(*n*) = *n* − 3

 **#11** (b) *f*: **N** → **Z** defined by *f*(*n*) = *n*2 + 4*n*

 **#11** (c) *f*: **Z** → **Z** defined by *f*(*n*) = *n*2 + 4*n*

**Cardinality**

**#12**. For each statement, is it True or False? (no explanation required)

    \_\_\_\_\_\_\_\_ (a)   $\bigcap\_{n=1}^{\infty } \left[5-\frac{1}{n}, 5+\frac{1}{n}\right] $ is a countable set.

 \_\_\_\_\_\_\_\_ (b)   $\bigcup\_{n=1}^{\infty }\left[5-\frac{1}{n}, 5+\frac{1}{n}\right]$ is a countable set.

 \_\_\_\_\_\_\_\_ (c)  Every subset of the rational numbers is countable.

  \_\_\_\_\_\_\_\_  (d)  Every set containing an irrational number is uncountable.

**#13**. State a specific function *f* that is a bijection *f*: (0, 1) → (1, ∞). (Note that you can then conclude that intervals (0, 1) and (1,∞) have the same cardinality. You are not asked to carry out a formal proof that your *f* is bijective.)

**#14**. HINT: Look at page 2 of my posted notes on Cardinality.

 **#14(a)** Show that the intervals (0, 1) and (3, 7) have the same cardinality by finding a

 bijection *f*:(0, 1) → (3, 7).

**#14(b)** Suppose *s* < *t*.

Prove that the intervals (0, 1) and (*s*, *t*) have the same cardinality by finding a bijection *f*:(0, 1) → (*s*, *t*).

 **#14(c)** Suppose *a* < *b* and *c* < *d*. Our goal is to show that open intervals (*a*, *b*) and (*c*, *d*) have the same cardinality. By part (b), there exist bijections *g*:(0, 1) → (*a*, *b*) and *h*:(0, 1) → (*c*, *d*).

State a **specific function *f*** that is a bijection *f*: (*a*, *b*) → (*c*, *d*), where *f* is an appropriate composition of functions involving functions *g*, *h*, and/or their inverses. [Recall composition of functions ---see Relations and Functions notes, page 9]. You do *not* need a complicated formula. Just make use some of the functions *g*, *h*, *g*−1, *h*−1, with an appropriate composition.

**ORDERED FIELDS**

**#15.** Consult the list of properties A1 - A5, M1 - M5, DL, O1 - O4 from my Ordered Field notes.

Rather than considering those properties applied on the set **R** of real numbers, restrict the set as indicated below. In other words, check which properties still apply, when **R** is replaced by the set specified.

 **#15 (a)** Which properties are *not* satisfied on the set **N**? (Just list the identifying labels.)

 **#15(b)** Which properties are *not* satisfied on the set **Z**? (Just list the identifying labels.)

 **#15(c)** Let *S* = {*x* in **R** such that *x* ≤ 0}.Which properties are *not* satisfied on the set *S*? (Just list the identifying labels.)

**#16.** Prove that 0 < 1/2 < 1. Fill in the blanks of the proof. **Refer to the field axioms and order axioms and the Theorem in my Ordered Field notes, pages 1-2.**

**Proof:**

Note that 0 < 1 (by Theorem part \_\_\_)

Adding 1 to both sides, 0 + 1 < 1 + 1 by order axiom \_\_\_.

0 + 1 = 1 by field axiom \_\_\_\_\_, and 1 + 1 = successor of 1, which is designated by 2 (Peano axiom in Induction notes).

So, we have 1 < 2.

Since 0 < 1 and 1 < 2, we have 0 < 1 < 2.

Then 0 < 1/2 < 1/1 by Theorem, part \_\_\_

Note that 1/1 = 1 (because 1 ⋅ 1 = 1).

Thus, we have 0 < 1/2 < 1, as desired.

**#17.** Let *r* be a real number.

 **Claim:** If 0 ≤ *r* < *ε* for all real numbers *ε* > 0, then *r* = 0.

**#17** (a) Write the negation of this sentence: 0 ≤ *r* < *ε* for all real numbers *ε* > 0.

Note that this sentence is: "0 ≤ *r* and *r* < *ε* for all real numbers *ε* > 0."

**#17** (b) Fill in the blanks of the proof of the claim. **Refer to the field axioms and order axioms and the Theorem in my Ordered Field notes, pages 1-2.**

**Proof of Claim, by contradiction:**

Suppose not.

Suppose for all real numbers *ε* > 0, we have 0 ≤ *r* < *ε* , but *r* ≠ 0.

Since 0 ≤ *r* and *r* ≠ 0, we must have *r* \_\_ 0. (fill in blank with <, = or, >, whichever is appropriate)

Set *ε* = (1/2) *r.*

Since 1/2 > 0 (by problem #16) ,

 we have *ε* = (1/2) *r* > (1/2) ⋅ 0 by order axiom \_\_\_\_.

 = 0 since (1/2) ⋅ 0 = 0 by Theorem, part \_\_\_.

Since 1/2 < 1 (by problem #16) and *r* \_\_ 0, we have *ε* = (1/2) *r* < 1 ⋅ *r* by order axiom \_\_\_\_,

 and so *ε* = (1/2) *r* < *r* since1 ⋅ *r* = *r* by field axiom \_\_\_\_.

In summary, we know *r* \_\_ 0 and we have found a particular ε > 0 for which *r* \_\_ *ε*. (fill in the blanks to show the correct order relationships between the numbers)

We have reached a contradiction of our hypothesis, and so we conclude that *r* must be equal to 0.